Recommended Reading: Munkres J.R. Topology

The current homework is on the last page

1. Determine which of the following statements are true for all sets A, B, C, D. If a double implication (equivalence) fails, determine whether at least one of the possible implications holds. If an equality fails, determine whether one or the other of the possible inclusions holds.

   (a) $A \supset C$ and $B \supset C \Leftrightarrow (A \cup B) \supset C$

   (b) $A \supset C$ or $B \supset C \Leftrightarrow (A \cup B) \supset C$

   (c) $A \supset C$ and $B \supset C \Leftrightarrow (A \cap B) \supset C$

   (d) $A \supset C$ or $B \supset C \Leftrightarrow (A \cap B) \supset C$

   (e) $A - (A - B) = B$

   (f) $A - (B - A) = A - B$

   (g) $A \cap (B - C) = (A \cap B) - (A \cap C)$

   (h) $A \cup (B - C) = (A \cup B) - (A \cup C)$

   (i) $(A \cap B) \cup (A - B) = A$

   (j) $A \subset C$ and $B \subset D \Leftrightarrow (A \times B) \subset (C \times D)$

   (k) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

   (l) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

   (m) $A \times (B - C) = (A \times B) - (A \times C)$

   (n) $(A - B) \times (C - D) = ((A \times C) - (B \times C)) - (A \times D)$

   (o) $(A \times B) - (C \times D) = (A - C) \times (B - D)$

2. Write each of the following subsets of $\mathbb{R} \times \mathbb{R}$ as a cartesian product of two subsets of $\mathbb{R}$ where possible.

   (a) $\{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{Z}\}$

   (b) $\{(x, y) \in \mathbb{R}^2 \mid \max\{|x|, 2|y|\} < 1\}$

   (c) $\{(x, y) \in \mathbb{R}^2 \mid 2x + 3y = 0\}$

   (d) $\bigcup_{n=1}^{\infty}\{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}$

3. Recall that a topology on a set X is a subset $T \subset \mathcal{P}(X)$ of the power set of X such that

   (a)
   $$\emptyset, X \in T\ ,$$

   (b)
   $$S \subset T \quad \Longrightarrow \quad \bigcup_{U \in S} U \in T \quad \text{and}$$

   (c)
   $$S \subset T \textbf{ finite} \quad \Longrightarrow \quad \bigcap_{U \in S} U \in T\ .$$

For each of the following subsets of $\mathcal{P}(\mathbb{R})$,

$$T_1 = \{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\emptyset, X\} \quad \text{and} \quad T_2 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, X\} ,$$

decide whether it is a topology on $\mathbb{R}$. Thus you either have to prove that T_i is a topology or show by a counterexample that at least one of the above properties (3a)-(3c) is violated.

please hand up 3 on Monday, 16/02 in class

4. Let X, Y be sets and $f \colon X \to Y$ be a map.
   (a) Let $\mathcal{B}$ be a basis for a topology of Y and consider the family
   $$f^{-1}(\mathcal{B}) := \{f^{-1}(B) \mid B \in \mathcal{B}\}$$
   of subsets of X. Prove that $f^{-1}(\mathcal{B})$ is a basis for a topology of X.
   Hint: $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$
   (b) Show (by a counterexample) that if $\mathcal{B}$ is a basis for a topology on X, then $f(\mathcal{B}) := \{f(B) \mid B \in \mathcal{B}\}$ need not be a basis for a topology on Y. What happens if f is surjective?

please hand up 4 on Thursday, 26/02 in class

5. For each of the following, either prove the statement or provide a counterexample:
   (a) If X is a set and A is a set of topologies on X, then $\bigcup_{T \in A} T$ is a topology on X
   (b) If X is a set and A is a set of topologies on X, then $\bigcap_{T \in A} T$ is a topology on X
   (c) A subset of a topological space is either open or closed.
   (d) The intersection of any family of closed sets of a topological space is closed.
   (e) Let T, T' be topologies on a set X such that $T \subset T'$.
      i. If (X, T) is connected then (X, T') is connected.
      ii. If (X, T') is connected then (X, T) is connected.

6. If $A \subset X$ are sets, we call the map $i \colon A \to X$ sending $a \mapsto a$ the inclusion. Let X be a topological space and $A \subset X$ be a subspace. A retraction of X onto A is a continuous left inverse of i, i.e. a continuous map $r \colon X \to A$ such that $r \circ i = \mathbf{id}_A$. Is there a retraction of the interval $I = [0, 1]$ (carrying the standard topology) to the boundary $\partial I = \{0, 1\}$?

please hand up 5-6 Wednesday, 11/3 in class

7. Do the following problems in the handout:
   pp14-15 Problems 4,5,7,8
   p20 Problem 1
   p44 problems 1 and 3
   Hint: *The notation in class for X^ω was $X^{\mathbb{N}}$. Thus you have to find a bijection of $\{0,1\}^{\mathbb{N}}$ onto a proper subset of $\{0,1\}^{\mathbb{N}}$.*
   p51 Problem 5

8. Find an example of a compact topological space (X, T) and a compact subset $A \subset X$ which is not closed.

9. Let T be the standard topology on $\mathbb{R}$. For a sequence $u \in T^{\mathbb{N}}$ of open sets in $\mathbb{R}$, denote by
   $$U_u := \{a \in \mathbb{R}^{\mathbb{N}} \mid \forall n : a(n) \in u(n)\}$$
   The box topology on $\mathbb{R}^{\mathbb{N}}$ is the topology T_b whose basis is
   $$\mathcal{B} = \{U_u \mid u \in T^{\mathbb{N}}\} .$$
   Is $(\mathbb{R}^{\mathbb{N}}, T_b)$ connected?
   Hint: *Look at the sets of bounded resp. unbounded sequences!*

10. Prove or provide a counterexample to the following statement: *"If X is a topological space and $A \subset X$ connected, then the interior of A is connected."*

    Hint: *The interior A° of a subset A of a topological space (X, T) is*

$$A^\circ = \bigcup_{U \subset A, U \in T} U \ .$$

11. Find a metric space X and a bounded closed subset $A \in X$ such that A is not compact.

    Hint: *A subset A of a metric space (X, d) is bounded, if there is $r \in \mathbb{R}^+$ and $p \in X$ such that $A \in B_r(p)$.*

12. Let X, Y be topological spaces, Y compact, and let $p \colon X \times Y \to X$, be the projection onto the X-factor, i.e. such that $p(x, y) = x$ for all $x \in X$, $y \in Y$. Prove that p is closed.

    Hint: *A map $f \colon X \to Y$ of topological spaces X, Y is closed, if $f(A) \subset Y$ is closed whenever $A \subset X$ is closed.*

    please hand up 8-12 Wednesday, 25/3 in class

13. Consider $\mathbb{R}^2$ with the standard topology. For each of the sets A_i given below decide whether they are closed, open, compact, dense, connected. Also determine for each of the A_i the interior A°, the closure $\overline{A}$ and the boundary ∂A.

    Hint: *The boundary ∂A of a subset $A \subset X$ of a topological space X is $\partial A = \overline{A} - A^\circ$.*

    (a) $A_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x = y\}$,
    (b) $A_2 = \{(x, y) \in \mathbb{R}^2 \,|\, x \in \mathbb{Q},\ y = 0\}$,
    (c) $A_3 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 23\}$,
    (d) $A_3 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x^2 - y^2 \leq 1\}$.

14. Let (X, d) be a metric space and $f \colon X \to X$ a contraction, i.e. a map such that there is $\alpha \in [0, 1)$ such that

$$\forall x, y \in X : d(f(x), f(y)) \leq \alpha d(x, y) \ .$$

    (a) Prove that f has a unique fixed point, i.e. a point $x \in X$ with $f(x) = x$ if X is compact.
    (b) Find a contraction $f \colon [1, 2] \to [1, 2]$ (wrt the standard metric) whose fixed point is $\sqrt{2}$.

15. Let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Q} \ .$$

    Let $X = \mathbb{R}/\sim$ endowed with the quotient topology. Is X connected, compact, metric, Hausdorff?
    please hand up 13-15 Wednesday, 8/4 in class

16. This problem is for those who feel uncomfortable about formalism. Please write down the definitions of the following notions in two ways: firstly as a formula, secondly as precisely as possible in words, using as few as possible mathematical symbols. Use the following example as a guideline:

notion to be defined: **Topology on a Set**

$$T \text{ is a topology on the set } X \quad \overset{\text{def}}{\Longleftrightarrow} \quad \left\{ \begin{array}{ccc} & T \subset \mathcal{P}(X) & \\ & \emptyset, X \in T & \\ Z \subset T & \Longrightarrow & \bigcup_{W \in Z} W \in T \\ Z \subset T, \#Z < \infty & \Longrightarrow & \bigcap_{W \in Z} W \in T \end{array} \right\}$$

now in plain english:
A topology on a set X is a family of subsets of X with the following properties:
The empty set and X are elements of this family,
any union of elements of the family is an element of the family,
the intersection of finitely many elements of the family is an element of the family.

Write down definitions for: closed subset of a topological space, connected, compact, Hausdorff topological space, metric topology, quotient topology, product topology, basis for a topology, topology generated by a basis, closure, interior, boundary of a subset of a topological space.

17. Let X be a topological space and Y be a star-shaped subset of a normed vector space V. For continuous maps $f_0, f_1 \colon X \to Y$, give a homotopy between f_0 and f_1.

Hint: *If $c \in Y$ is the center of Y, then there are homotopies between any map $X \to Y$ and the constant map with value c. Compose two such homotopies.*

18. Consider the following families $T_i \subset \mathcal{P}(\mathbb{Q})$ of subsets of the set of rational numbers. Which of the T_i are topologies on $\mathbb{Q}$? For each i either prove that T_i is a topology or show by means of a counterexample which axiom for a topology is violated.

    (a) $T_a = \{\mathbb{Q} \cap (-r, r) \mid r \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{Q}\}$
    (b) $T_b = \{A \subset \mathbb{Q} \mid 0 \in A \text{ or } 1 \in A\} \cup \{\emptyset\}$
    (c) $T_c = \{\emptyset, \mathbb{Z}, \mathbb{N}, \mathbb{Q}\}$
    (d) $T_d = \{(a, b) \cap \mathbb{Q} \mid a, b \in \mathbb{R}, a < b, b^2 = a^2 + 1\} \cup \{\emptyset, \mathbb{Q}\}$

19. Let $\sim$ be the equivalence relation on R with $x \sim y \iff x - y \in \mathbb{Z}$. Endow $\mathbb{R}$ with the standard topolgy and consider $X = \mathbb{R}/\sim$ with the quotient topology. For the following subsets $A_i \subset X$ determine the closure $\overline{A_i}$, the interior A°, the boundary ∂A_i and state whether A_i is connected, compact. Below, we denote by $[x] = \{y \in \mathbb{R} \mid y \sim x\}$ the equivalence class of $x \in \mathbb{R}$.

    (a) $A_a = \{[1/n] \mid n \in \mathbb{N}\}$
    (b) $A_b = \{[1/2], [1/3]\}$
    (c) $A_c = \{[n/2^k] \mid n, k \in \mathbb{N}\}$
    (d) $A_d = \{[x] \mid 0 < x \leq 1\}$

20. Let $X.Y$ be topological spaces and $A \subset X$, $B \subset Y$ be closed subsets. Prove that $A \times B$ is a closed subset of $X \times Y$ with respect to the product topology.

21. Decide which of the following topological spaces (X_i, T_i) are connected/compact. Prove your assertion!

    (a) $X_a = \mathbb{Z}$, $T_a = \{\{x \in \mathbb{Z} \mid |x| \leq b\} \mid b \in \mathbb{N}\} \cup \{\emptyset, \mathbb{Z}\}$
    (b) $X_b = \mathbb{R}^7$, $T_b = \{A \subset \mathbb{R}^7 \mid 0 \in A\} \cup \{\emptyset\}$

(c) $X_c = [0,1]$, $T_c = \{A \subset [0,1] \mid [0,1] - A \text{ countable }\} \cup \{\emptyset\}$

please hand up 17-21 Wednesday, 22/4 in class

22. Prove that a disconnected space can not have a connected strong deformation retract.

    Hint: *A strong deformation retract of a topological space X is a subset $A \subset X$ such that there is a map $r\colon X \to A$ (the "retraction") with $r \circ \iota = \mathbf{id}_A$ and $\iota \circ r \simeq \mathbf{id}_X$ rel A. Here $\iota\colon A \to X, \iota\colon a \mapsto a$, denotes the inclusion of A in X.*

23. Is the open ball $B_1^n(0) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ a strong deformation retract of $\mathbb{R}^n$?

24. The cone over a topological space X is the quotient $CX = X \times [0,1]/\sim$ where the equivalence relation $\sim$ is given by

$$(x,1) \sim (x',1) \quad \text{for all} \quad x, x' \in X.$$

    Prove that the cone of any topological space is contractible.

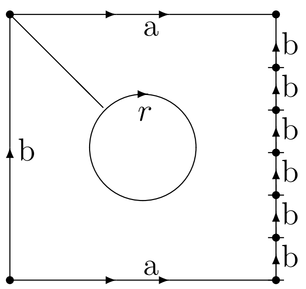
    Hint: *Prove that $\{X \times \{1\}\} \subset CX$ is a strong deformation retract.*

25. Let $X = S^1 \times [0,1]/\sim$ (with the quotient topology) where the equivalence relation $\sim$ is given by

$$(z,0) \sim (z^6,1) \quad \text{for} \quad z \in S^1 .$$

    Find a presentation for the fundamental group of X.

    Solution: Look at the following square



    The space X is obtained by glueing a square as indicated in the picture. Thus the the four corners and the five points on the right (the dots) are all identified to one point (the base point) and the two segments on top and bottom (denoted by a) and the one segment b on the left and the six segments b on the right are identified as indicated by the arrows. Thus the fundamental group of the whole space is generated by two loops arising from a and b. The loop $r = ab^{-6}a^{-1}b$ is once around the whole square and can be contracted along the line to the upper left corner to the base point. Thus the fundamental group of X is

$$\langle a, b \mid ab^{-6}a^{-1}b\rangle .$$

please hand up 22-25 Wednesday, 6/5 in class

26. Which of the following subsets T_i of the power set of $\mathbb{Z}^{\mathbb{N}}$ are topologies?

    (a) $T_1 = \{\{f \in \mathbb{Z}^{\mathbb{N}} \mid \forall n \in \mathbb{N} : f(n) < c\} \mid c \in \mathbb{R}^+\} \cup \{\emptyset, \mathbb{Z}^{\mathbb{N}}\}$,

    Solution: For $c \in \mathbb{R}$ let

$$U_c = \{f \in \mathbb{Z}^{\mathbb{N}} \mid \forall n \in \mathbb{N} : f(n) < c\} .$$

    Consider the union

$$B := \bigcup_{c \in \mathbb{R}^+} U_c = \{f \in \mathbb{Z}^{\mathbb{N}} \mid \exists c \in \mathbb{R}^+ \forall n \in \mathbb{N} : f(n) < c\} .$$

    This is the set of all functions bounded from above. Clearly $B \neq \emptyset$, $B \neq \mathbb{Z}^{\mathbb{N}}$ and there is no $c \in \mathbb{R}$ such that $B = U_c$. Hence $B \notin T_1$.

(b) $T_2 = \{\{f \in \mathbb{Z}^{\mathbb{N}} \mid f(n) = 0\} \mid n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{Z}^{\mathbb{N}}\}$,
Solution: For $n \in \mathbb{N}$ let $U_n := \{f \in \mathbb{Z}^{\mathbb{N}} \mid f(n) = 0\}$. Thus

$$T_2 = \{U_n \mid n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{Z}^{\mathbb{N}}\} .$$

Now T_2 is not a topology since, for instance, $U_1 \cap U_2 = \{f \colon \mathbb{N} \to \mathbb{Z} \mid f(1) = f(2) = 0\} \neq U_n$ for any $n \in \mathbb{N}$.

(c) $T_3 = \{\{f \in \mathbb{Z}^{\mathbb{N}} \mid f(0) \in U\} \mid U \subset \mathbb{Z}\} \cup \{\emptyset, \mathbb{Z}^{\mathbb{N}}\}$.
Solution: T_3 is a topology. Clearly, $\emptyset, \mathbb{Z}^{\mathbb{N}} \in T_3$. For $U \subset \mathbb{Z}$ define

$$V_U := \{f \colon \mathbb{N} \to \mathbb{Z} \mid f(0) \in U\} .$$

Now let $A \subset T_3$. If $\mathbb{Z}^{\mathbb{N}} \in A$ then $\bigcup_{a \in A} a = \mathbb{Z}^{\mathbb{N}} \in T_3$. Otherwise there is a subset $P \subset \mathcal{P}(\mathbb{Z})$ such that

$$A = \{V_U \mid U \in P\} .$$

Thus

$$\bigcup_{a \in A} a = \bigcup_{U \in \mathcal{P}} V_U = V_{\bigcup_{U \in \mathcal{P}} U} \in T_3 .$$

For the intersection axiom simply exchange the roles of $\mathbb{Z}^{\mathbb{N}}$ and $\cup$ with those of $\emptyset$ and $\cap$ in the above.

27. Let X, Y be path connected topological spaces and assume $f \colon X \to Y$ is a homeomorphism. Prove that X is simply connected if and only if Y is simply connected.

Hint: *A topological space X is simply connected if X is path connected and any loop $f \colon [0,1] \to X$ is homotopic* (rel$\{0,1\}$) *to the constant loop at $f(0) = f(1)$.*

Solution: If $\omega \colon [0,1] \to Y$ is a loop and X is simply connected, then there is a homotopy $H \colon [0,1] \times [0,1] \to X$ of loops $f^{-1} \circ \omega \simeq_H c_{f^{-1}(\omega(0))}$, where c_x denotes the constant loop at x. Then the composition $f \circ H$ gives a homotopy $\omega \simeq_{f \circ H} c_{\omega(0)}$. For the converse, apply the same argument with X, Y interchanged.

28. Let $A \subset \mathbb{R}^n$, $n > 1$, be a nonempty connected open subset, and $x_1, \ldots, x_r \in A$ be finitely many points. Prove that $A - \{x_1, \ldots, x_r\}$ is connected.

29. Consider the set $X = \mathbb{Q}^{\mathbb{Q}}$ with the product topology. Thus a set $U \subset \mathbb{Q}^{\mathbb{Q}}$ is open if for each $f \in U$, $f \colon \mathbb{Q} \to \mathbb{Q}$, there is a finite set $\{q_1, \ldots, q_r\} \subset \mathbb{Q}$ and $\epsilon \in \mathbb{R}$ such that

$$\{g \colon \mathbb{Q} \to \mathbb{Q} \mid |g(q_i) - f(q_i)| < \epsilon\} \subset U .$$

Hint: *Prove this!* For the following subsets $A_i \subset X$ determine the closure $\overline{A_i}$, the interior A°, the boundary ∂A_i. Say whether A_i is connected or not and whether A_i is compact or not.

(a) $A_a = \{f \colon \mathbb{Q} \to \mathbb{Q} \mid \forall n \in \mathbb{N} : f(n) < 1\}$
Solution: $A_a^\circ = \emptyset$, $\overline{A_a} = \{f \colon \mathbb{Q} \to \mathbb{Q} \mid \forall n \in \mathbb{N} : f(n) \leq 1\}$, $\partial A_a = \overline{A_a}$. A_a is not connected and not compact.

(b) $A_b = \{f \colon \mathbb{Q} \to \mathbb{Q} \mid f(0) \in \mathbb{Z}\}$
Solution: $A_b^\circ = \emptyset$, $\overline{A_b} = A_b = \partial A_b$. A_b is not connected and not compact.

(c) $A_c = \{f \colon \mathbb{Q} \to \mathbb{Q} \mid \forall n \in \mathbb{N} : f(n) = 0\}$
Solution: $A_c^\circ = \emptyset$, $\overline{A_c} = \partial A_c = A_c$. A_c is not connected and not compact.

(d) $A_d = \mathbb{Q}[x] = \{f \colon \mathbb{Q} \to \mathbb{Q} \mid \exists c_0, \ldots c_r \in \mathbb{Q} \forall x \in \mathbb{Q} : f(x) = c_0 + c_1 x + c^2 x^2 + \cdots + c_r x^r\}$
Solution: $A_d^\circ = \emptyset$, $\overline{A_d} = \partial A_d = X$. A_d is not connected and not compact.

Use Problems 26-29 to practise in the study week!