

ADMISSIBLE MEASURES IN ONE DIMENSION

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ABSTRACT. In this note we show that p -admissible measures in one dimension (i.e. doubling measures admitting a p -Poincaré inequality) are precisely the Muckenhoupt A_p -weights.

1. INTRODUCTION

In the last two decades, it has been observed that much of the theory for p -harmonic functions can be extended to the situation when the Lebesgue measure on \mathbf{R}^n is replaced by another measure satisfying certain conditions, see e.g. Fabes–Kenig–Serapioni [2] and Heinonen–Kilpeläinen–Martio [4]. More precisely, Theorem 2 in Hajlasz–Koskela [3] and Theorem 5.2 in Heinonen–Koskela [5] show that the following two conditions are exactly what is needed for the theory to go through.

Definition 1.1. A measure μ on \mathbf{R}^n is called p -admissible with $p \geq 1$, if it satisfies the following two conditions.

- It is *doubling*, i.e. there is a constant $C > 0$ such that

$$\mu(2B) < C\mu(B)$$

for all balls $B \subset \mathbf{R}^n$, where $2B$ denotes the ball concentric with B and with twice the radius.

- It admits the *weak p -Poincaré inequality*, i.e. there exist $C > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq Cr \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} |\nabla u|^p d\mu \right)^{1/p}$$

holds whenever B is a ball with radius r and u is, say, a locally Lipschitz function on λB . Here and in what follows, $u_B = \mu(B)^{-1} \int_B u d\mu$.

The Hölder inequality implies that every p -admissible measure is also p' -admissible for all $p' > p$. Conversely, by a recent result due to Keith–Zhong [6], every p -admissible measure with $p > 1$ is also p' -admissible for some $p' < p$.

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Unfortunately, in many situations, the Poincaré inequality is rather difficult to verify. In this note we give a more straightforward characterization of admissible measures in one dimension, namely we prove the following result.

Theorem 1.2. *Let μ be a measure on \mathbf{R} and $p \geq 1$. Then μ is p -admissible in \mathbf{R} if and only if $d\mu = w dx$ and w is a Muckenhoupt A_p -weight.*

Definition 1.3. A nonnegative function w on \mathbf{R}^n is a Muckenhoupt A_p -weight with $p \geq 1$, if for some $C > 0$ and all balls $B \subset \mathbf{R}^n$,

$$\frac{1}{|B|} \int_B w dx < \begin{cases} C \left(\frac{1}{|B|} \int_B w^{1/(1-p)} dx \right)^{1-p} & \text{for } p > 1, \\ C \operatorname{ess\,inf}_B w & \text{for } p = 1, \end{cases}$$

where $|B|$ denotes the Lebesgue measure of B .

Remark 1.4. Note that Theorem 1.2 fails in \mathbf{R}^n if $n \geq 2$. By e.g. Corollary 15.35 in Heinonen–Kilpeläinen–Martio [4], the measures $d\mu = |x|^\alpha dx$ with $\alpha > 0$ are p -admissible in \mathbf{R}^n , $n \geq 2$, for all $p > 1$, but belong to A_p if and only if $p > 1 + n\alpha$.

To prove Theorem 1.2, we use the following lemma. For a proof, see the corollary on p. 200 in Stein [7].

Lemma 1.5. *Let μ be a nonnegative Borel measure on \mathbf{R}^n and assume that there exists $C > 0$ such that*

$$\frac{1}{|B|} \int_B f(x) dx \leq C \left(\frac{1}{\mu(B)} \int_B f^p d\mu \right)^{1/p}$$

for all balls $B \subset \mathbf{R}^n$ and all nonnegative measurable functions f on B . Then μ is absolutely continuous with respect to the Lebesgue measure, $d\mu = w dx$ and w is a Muckenhoupt A_p -weight.

In the rest of this note, $C > 0$ denotes a constant whose value may vary with each usage but depends only on the doubling constant of μ and on the constants in the Poincaré inequality.

Proof of Theorem 1.2. The “if” part of the theorem is proved e.g. in Theorem 15.21 in Heinonen–Kilpeläinen–Martio [4]. To prove the “only if” part, let $f \geq 0$ be a measurable function supported on an interval $I \subset \mathbf{R}$. For $k \in \mathbf{N}$, let $f_k = \min\{f, k\}$ and

$$u_k(x) = \int_{-\infty}^x f_k(t) \chi_I(t) dt.$$

Then u_k is Lipschitz and we can test the weak p -Poincaré inequality with it on the concentric double $2I$ of I . On the right-hand side we have

$$C|I| \left(\frac{1}{\mu(2\lambda I)} \int_{2\lambda I} (u'_k)^p d\mu \right)^{1/p} \leq C|I| \left(\frac{1}{\mu(I)} \int_I f^p d\mu \right)^{1/p}.$$

To estimate the left-hand side in the Poincaré inequality, let I_- and I_+ denote the parts of $2I \setminus I$ lying to the left and to the right of I , respectively. Then $u_k = 0$ on I_- and

$$u_k = \int_I f_k(x) dx$$

on I_+ . Using the doubling property of μ , the left-hand side in the Poincaré inequality can be estimated as

$$\begin{aligned} \frac{1}{\mu(2I)} \int_{2I} |u_k - (u_k)_{2I}| d\mu &\geq \frac{1}{\mu(2I)} \left(\int_{I_-} (u_k)_{2I} d\mu + \int_{I_+} \left(\int_I f_k(x) dx - (u_k)_{2I} \right) d\mu \right) \\ &\geq C \int_I f_k(x) dx. \end{aligned}$$

Inserting both estimates into the weak p -Poincaré inequality, together with the monotone convergence theorem, shows that the assumptions in Lemma 1.5 are satisfied and hence $d\mu = w dx$ with w a Muckenhoupt A_p -weight. \square

Remark 1.6. If we knew a priori that μ is absolutely continuous with respect to the Lebesgue measure, then Theorem 1.2 could also be obtained after some calculation from Theorem 1.4 in Chua–Wheeden [1].

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